

# A PROPERTY OF THE BROWN-YORK MASS IN SCHWARZSCHILD MANIFOLDS

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ABSTRACT. We will extend partially our previous results about the limit of the Brown-York mass of a family of convex revolution surfaces in the Schwarzschild manifold such that these surfaces may have unbounded ratios of their radii.

## 1. INTRODUCTION

In this paper, we will continue to study the limiting behavior of the Brown-York mass of a family of convex revolution surfaces in the Schwarzschild manifold and extend our previous results in [8]. Throughout this paper, we will denote  $(\mathbb{R}^3, \delta_{ij})$  as the 3-dimensional Euclidean space,  $x^1, x^2, x^3$  as the standard coordinates of  $\mathbb{R}^3$ ,  $r$  and  $\partial$  as the Euclidean distance and the standard derivative operator on  $\mathbb{R}^3$  respectively. For the sake of convenience, let us first recall some definitions. First of all, we will adopt the following definition of asymptotically flat manifolds.

**Definition 1.1.** *A complete three dimensional manifold  $(M, \lambda)$  is said to be asymptotically flat (AF) of order  $\tau$  (with one end) if there is a compact subset  $K$  such that  $M \setminus K$  is diffeomorphic to  $\mathbb{R}^3 \setminus B_R(0)$  for some  $R > 0$  and in the standard coordinates in  $\mathbb{R}^3$ , the metric  $\lambda$  satisfies:*

$$(1.1) \quad \lambda_{ij} = \delta_{ij} + \sigma_{ij}$$

with

$$(1.2) \quad |\sigma_{ij}| + r|\partial\sigma_{ij}| + r^2|\partial\partial\sigma_{ij}| + r^3|\partial\partial\partial\sigma_{ij}| = O(r^{-\tau}),$$

for some constant  $1 \geq \tau > \frac{1}{2}$ .

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A coordinate system of  $M$  near infinity so that the metric tensor in this system satisfy the above decay conditions is said to be admissible. In such a coordinate system, we can define the ADM mass as follows.

**Definition 1.2.** *The Arnowitt-Deser-Misner (ADM) mass (see [1]) of an asymptotically flat manifold  $(M, \lambda)$  is defined as:*

$$(1.3) \quad m_{ADM}(M, \lambda) = \lim_{r \rightarrow \infty} \frac{1}{16\pi} \int_{S_r} (\lambda_{ij,i} - \lambda_{ii,j}) \nu^j d\Sigma_r^0,$$

where  $S_r$  is the Euclidean sphere,  $d\Sigma_r^0$  is the volume element induced by the Euclidean metric,  $\nu$  is the outward unit normal of  $S_r$  in  $\mathbb{R}^3$  and the derivative is the ordinary partial derivative.

We always assume that the scalar curvature is in  $L^1(M)$  so that the limit exists in the definition. In [2], Bartnik showed that the ADM mass is a geometric invariant.

On the other hand, there have been many studies on the relation between the ADM mass of an AF manifold and the so called quasi-local mass. Let us recall the definition of the Brown-York quasi-local mass. Suppose  $(\Omega, \mu)$  is a compact three dimensional manifold with smooth boundary  $\partial\Omega$ , if moreover  $\partial\Omega$  has positive Gauss curvature, then the Brown-York mass of  $\partial\Omega$  is defined as (see [5, 6]):

**Definition 1.3.**

$$(1.4) \quad m_{BY}(\partial\Omega) = \frac{1}{8\pi} \int_{\partial\Omega} (H_0 - H) d\sigma$$

where  $H$  is the mean curvature of  $\partial\Omega$  with respect to the outward unit normal and the metric  $\mu$ ,  $d\sigma$  is the volume element induced on  $\partial\Omega$  by  $\mu$  and  $H_0$  is the mean curvature of  $\partial\Omega$  when embedded in  $\mathbb{R}^3$ .

The existence of an isometric embedding in  $\mathbb{R}^3$  for  $\partial\Omega$  was proved by Nirenberg [17], the uniqueness of the embedding was given by [13, 19, 18], so the Brown-York mass is well-defined.

It can be proved that the Brown-York mass and the Hawking quasi-local mass [11] of the coordinate spheres tends to the ADM mass in some AF manifolds, see [6, 12, 4, 3, 22, 9], even of nearly round surfaces [21], and of a family of convex revolution surfaces in an asymptotically Schwarzschild manifold [8] for the Brown-York mass. The ratios of the radii of these surfaces are all bounded. In this paper, we will generalize some results in [8] partially in that we allow the ratios of the radii of the family of surfaces to be unbounded. We will consider a kind of AF manifolds, called Schwarzschild manifolds, defined as follows:

**Definition 1.4.**  $(N, g)$  is called a *Schwarzschild manifold* if  $N = \mathbb{R}^3 \setminus K$ ,  $K$  is a compact set containing the origin, and in the standard coordinates of  $\mathbb{R}^3$ ,

$$g_{ij} = \phi^4 \delta_{ij}, \phi = 1 + \frac{2m}{r}, m > 0.$$

Clearly,  $(N, g)$  is an AF manifold and the scalar curvature of  $(N, g)$  is zero [15] (page 283). Moreover, the ADM mass is equal to  $m$ .

Let  $w(\varphi), z(\varphi)$  be smooth functions on  $[0, l]$  such that the surface of revolution generated by  $w$  and  $z$ :

$$(1.5) \quad (w(\varphi) \cos \theta, w(\varphi) \sin \theta, z(\varphi))$$

is a smooth convex surface diffeomorphic to  $\mathbb{S}^2$  and

$$(1.6) \quad \begin{cases} C_1^2 \leq w^2 + z^2 \leq C_2^2, \text{ for } C_1, C_2 > 0 \\ w'^2 + z'^2 = 1 \\ w \geq 0 \text{ on } [0, l] \text{ and } z(0) > z(l). \end{cases}$$

Let  $f(a)$  be a function such that  $f(a) \geq 1$  for all  $a \geq 1$ . We define the family of surfaces  $S_a$  by the parametrization

$$(1.7) \quad (aw(\varphi) \cos \theta, aw(\varphi) \sin \theta, ah_a(\varphi))$$

where  $h_a(\varphi) = f(a)z(\varphi)$ . Note that  $S_a$  forms an exhaustion of  $N$  as  $a \rightarrow \infty$ .

We will prove the following:

**Theorem 1.1.** *Suppose  $(S_a, g|_{S_a})$  has positive Gaussian curvature, then the Brown York mass of  $S_a$  tends to the ADM mass of  $(N, g)$ . That is*

$$\lim_{a \rightarrow \infty} m_{BY}(S_a) = m.$$

One example of surfaces satisfying the conditions in Theorem 1.1 is the family of ellipsoids:

$$S_a = \left\{ (x^1)^2 + (x^2)^2 + \frac{2m(x^3)^2}{a} = a^2 \right\}$$

which has unbounded ratios of their radii as  $a \rightarrow \infty$ .

From Theorem 1.1, we have

**Corollary 1.1.** *Suppose  $\frac{f^2}{a} = o(1)$  for sufficiently large  $a$ , then*

$$\lim_{a \rightarrow \infty} m_{BY}(S_a) = m.$$

Clearly the above example shows that the condition  $\frac{f^2}{a} = o(1)$  is not necessary.

This paper is organized as follows. In Section 2, we will prove Theorem 1.1. Corollary 1.1 will be proven in Section 3.

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## 2. PROOF OF THEOREM 1.1

Let us first prove some lemmas. We can assume  $w$  is anti-symmetric about 0 and  $l$ ,  $z$  is symmetric about 0 and  $l$ . This implies

$$(2.1) \quad w(0) = w(l) = z'(0) = z'(l) = 0.$$

The Gaussian curvature  $\overline{K}$  of (1.5) with respect to  $\delta$  is ([7] p.162)

$$\overline{K} = \frac{z'(w'z'' - w''z')}{w} \text{ for } \varphi \in (0, l).$$

So by (1.6),  $z' < 0$  on  $(0, l)$ .

We will sometimes regard  $\phi$  as function of  $\varphi$  (by restricting on  $S_a$ ) by abuse of notation. We define

$$(2.2) \quad D = \sqrt{w'^2 + h'^2} = \sqrt{w'^2 + f^2 z'^2}.$$

Similar to Lemma 2.1 in [8], we have

**Lemma 2.1.** *The functions  $\frac{w}{z'}$  and  $\frac{w''}{z'}$  can be extended continuously to  $[0, l]$ .*

*Proof.* The Gaussian curvature of the surface given by (1.5) at  $(0, 0, z(0))$  is  $z''(0)^2 > 0$ , so  $\lim_{\varphi \rightarrow 0} \frac{w}{z'} = \frac{w'(0)}{z''(0)} < \infty$ . The case for  $\varphi = l$  is the same.

The plane curve  $(w(\varphi), z(\varphi))$  has curvature  $k = \frac{w''z' - z''w'}{(w', z') \cdot (-z'', w')}$ . Since  $(w'', z'') \perp (w', z')$  and  $(w'', z'') \perp (-z'', w')$ , we have  $w'' = kz'$ . From this we see that  $\frac{w''}{z'}$  can be extended to  $k(0)$  at  $\varphi = 0$ . The case for  $\varphi = l$  is the same.  $\square$

**Lemma 2.2.** *The following functions can be extended continuously to  $[0, l]$  such that*

$$(2.3) \quad \begin{aligned} \frac{w}{h'} &= O\left(\frac{1}{f}\right), \frac{w''}{h'} = O\left(\frac{1}{f}\right), \\ D' &= O\left(\frac{f^2}{D}\right), \frac{D'}{h'} = O\left(\frac{f}{D}\right) \end{aligned}$$

$$(2.4) \quad \phi' = O\left(\frac{fa}{r^2}\right), \frac{\phi'}{h'} = O\left(\frac{fa^2}{r^3}\right), \phi'' = O\left(\frac{f^2a^2}{r^3}\right).$$

*Proof.* The first line of (2.3) follows from Lemma 2.1. As  $D' = \frac{w'w'' + f^2z'z''}{D}$ ,  $|D'| \leq \frac{Cf^2}{D}$ . Also, using the fact that  $\frac{w}{h'} = O(f^{-1}) = O(1)$ ,  $|\frac{D'}{h'}| = |\frac{1}{D}(w'\frac{w''}{h'} + fz'')| = O(\frac{f}{D})$ . For (2.4), we have  $\phi' = -\frac{ma^2}{2r^3}(ww' + f^2zz')$ . So

$$\begin{aligned} |\phi'| &\leq \frac{ma^2}{2r^3}(w^2 + f^2z^2)^{\frac{1}{2}}(w'^2 + f^2z'^2)^{\frac{1}{2}} = \frac{mar}{2r^3}(w'^2 + f^2z'^2)^{\frac{1}{2}} \\ &= O\left(\frac{fa}{r^2}\right). \end{aligned}$$

Using (2.3) again,

$$\left|\frac{\phi'}{h'}\right| = -\frac{ma^2}{2r^3}\left(w'\frac{w}{h'} + fz\right) = O\left(\frac{fa^2}{r^3}\right).$$

Finally,

$$\begin{aligned} |\phi''| &= \left| \frac{3m}{2}r^{-5}a^4(ww' + hh')^2 - \frac{m}{2}r^{-3}a^2((w')^2 + (h')^2 + ww'' + hh'') \right| \\ &\leq \frac{3m}{2}r^{-5}a^4(w^2 + h^2)((w')^2 + (h')^2) \\ &\quad + \frac{m}{2}r^{-3}a^2(((w')^2 + (h')^2) + (w^2 + h^2)^{\frac{1}{2}}((w'')^2 + (h'')^2)^{\frac{1}{2}}) \\ &= \frac{3m}{2}r^{-3}a^2((w')^2 + (h')^2) + O(r^{-3}a^2f^2) = O(r^{-3}a^2f^2). \end{aligned}$$

Hence Lemma 2.2 holds.  $\square$

We want to compute the mean curvatures. By Lemma 2.4 in [8], we have

**Lemma 2.3.** *For a smooth revolution surface in  $(\mathbb{R}^3, \delta)$  parametrized by*

$$(2.5) \quad (au(\varphi) \cos \theta, au(\varphi) \sin \theta, av(\varphi)), \quad 0 < \varphi < l, 0 < \theta < 2\pi,$$

*its mean curvature  $\overline{H}$  with respect to  $\delta$  is*

$$(2.6) \quad \overline{H} = \frac{u''}{aTv'} - \frac{T'u'}{aT^2v'} - \frac{v'}{aTu} \quad \text{where } T = \sqrt{u'^2 + v'^2}.$$

Similar to Lemma 2.5 in [8], we can get

**Lemma 2.4.** *The mean curvature  $H$  of  $S_a$  with respect to  $g$  is*

$$H = \frac{w''}{a\phi^2 Dh'} - \frac{D'w'}{a\phi^2 D^2 h'} - \frac{h'}{a\phi^2 Dw} + 4\phi^{-3}n(\phi)$$

*where  $n$  is the outward unit normal vector of  $S_a$  with respect to  $\delta$ .*

*Proof.* It is similar to the proof of Lemma 2.5 in [8]. For completeness, we sketch it here. By Lemma 2.4, the mean curvature of  $S_a$  with respect to  $\delta$  is

$$(2.7) \quad \bar{H} = \frac{w''}{aDh'} - \frac{D'w'}{aD^2h'} - \frac{h'}{aDw}.$$

The mean curvature  $H$  of  $S_a$  with respect to  $g$  is ([20], page 72):

$$(2.8) \quad H = \phi^{-2} (\bar{H} + 4\phi^{-1}n(\phi))$$

where  $n$  is the outward unit normal vector of  $S_a$  with respect to  $\delta$ . Submitting (2.7) to (2.8), we can get Lemma 2.4.  $\square$

**Lemma 2.5.** *Suppose  $(S_a, g|_{S_a})$  has positive Gaussian curvature such that it can be uniquely isometrically embedded into  $\mathbb{R}^3$  (for sufficiently large  $a$ ), then the embedding is given by*

$$(2.9) \quad (au(\varphi) \cos \theta, au(\varphi) \sin \theta, av(\varphi)), \varphi \in [0, l], \theta \in [0, 2\pi]$$

where

$$(2.10) \quad u = \phi^2 w \quad \text{and} \quad v' = \phi^2 h' \left( 1 - \frac{4\phi'ww'}{\phi h'^2} - \frac{4\phi'^2 w^2}{\phi^2 h'^2} \right)^{\frac{1}{2}}.$$

*Proof.* The proof is similar to that of Lemma 2.6 in [8]. In  $(\varphi, \theta)$  coordinates, the metric on  $S_a$  induced by  $g$  can be written as:

$$(2.11) \quad ds^2 = a^2 \phi^4 (w'^2 + h'^2) d\varphi^2 + a^2 \phi^4 w^2 d\theta^2.$$

We can regard  $(S_a, ds^2)$  as the sphere with the metric  $ds^2$ . We want to find two functions  $u, v$  such that the surface written as the form (2.9) is an embedded surface  $S_a^e$  of  $S_a$  into  $(\mathbb{R}^3, \delta)$ . First of all, the induced metric by the Euclidean metric on the surface which is of the form (2.9) can be written as:

$$ds_e^2 = a^2 (u'^2 + v'^2) d\varphi^2 + a^2 u^2 d\theta^2.$$

Comparing this with (2.11), one can choose

$$(2.12) \quad u = \phi^2 w \quad \text{and} \quad u'^2 + v'^2 = \phi^4 D^2.$$

Consider

$$(2.13) \quad \begin{aligned} \phi^4 (w'^2 + h'^2) - u'^2 &= \phi^2 (\phi^2 (w'^2 + h'^2) - (2\phi'w + \phi w')^2) \\ &= \phi^2 (\phi^2 h'^2 - 4\phi\phi'ww' - 4\phi'^2 w^2) \\ &= \phi^4 h'^2 \left( 1 - \frac{4\phi'ww'}{\phi h'^2} - \frac{4\phi'^2 w^2}{\phi^2 h'^2} \right). \end{aligned}$$

By Lemma 2.2, the functions  $\frac{\phi'ww'}{\phi h'^2}, \frac{\phi'^2w^2}{\phi^2h'^2}$  can be extended continuously on  $[0, l]$  with  $\frac{\phi'ww'}{\phi h'^2} = O(a^{-1}), \frac{\phi'^2w^2}{\phi^2h'^2} = O(a^{-2})$ . So  $1 - \frac{4\phi'ww'}{\phi h'^2} - \frac{4\phi'^2w^2}{\phi^2h'^2} > 0$  for sufficiently large  $a$ . For these  $a$ , we can take

$$v' = \phi^2 h' \left( 1 - \frac{4\phi'ww'}{\phi h'^2} - \frac{4\phi'^2w^2}{\phi^2h'^2} \right)^{\frac{1}{2}},$$

so that  $u'^2 + v'^2 = \phi^4(w'^2 + h'^2)$ . Note that  $v'$  is an odd function for  $\varphi \in [-l, l]$ . By choosing an initial value, one can get an even function  $v$ . By the above argument, one has

$$v' = \phi^2 h' \left( 1 - \frac{2\phi'ww'}{h'^2} + O(a^{-2}) \right).$$

From (2.12) and (2.13), near  $\varphi = 0$ ,  $u, v$  can be extended naturally to  $(-\varepsilon, \varepsilon)$  for some  $\varepsilon > 0$ . Since  $u$  is an odd function in  $\varphi$ ,  $v$  is an even function in  $\varphi$ , and  $u'^2 + v'^2 = T^2 > 0$ , the generating curve in  $\{x^2 = 0\}$  is symmetric with respect to  $x^3$ -axis, and is smooth at  $\varphi = 0$ . Similarly, it is also smooth at  $\varphi = l$ . Hence the revolution surface determined by the choice of  $u, v$  as above can be extended smoothly to a closed revolution surface, which is the embedded surface of  $S_a$ .  $\square$

Now we are ready to prove Theorem 1.1.

*Proof of Theorem 1.1.* Let  $F = \phi^2 D$ , then by Lemma 2.3 and Lemma 2.4,

$$(2.14) \quad \begin{aligned} H_0 - H &= \left( \frac{u''}{aFv'} - \frac{w''}{a\phi^2 Dh'} \right) - \left( \frac{F'u'}{aF^2 v'} - \frac{D'w'}{a\phi^2 D^2 h'} \right) \\ &\quad - \left( \frac{v'}{aFu} - \frac{h'}{a\phi^2 Dw} \right) - 4\phi^{-3}n(\phi). \end{aligned}$$

By Lemma 2.2, we have

$$(2.15) \quad \frac{u''}{aFv'} - \frac{w''}{a\phi^2 Dh'} = \frac{4\phi'w'}{aDh'} + \frac{2\phi''w}{aDh'} + \frac{2\phi'ww''}{aDh'^3} + O\left(\frac{fa^3}{Dr^6}\right),$$

$$(2.16) \quad \begin{aligned} -\frac{F'u'}{aF^2 v'} + \frac{D'w'}{a\phi^2 D^2 h'} &= -\frac{2\phi'D'w}{aD^2 h'} - \frac{2\phi'w'}{aDh'} - \frac{2\phi'ww'^2 D'}{aD^2 h'^3} \\ &\quad + O\left(\frac{f^2 a^2 w}{D^3 r^5}\right) + O\left(\frac{fa^3}{Dr^6}\right), \end{aligned}$$

and

$$(2.17) \quad -\frac{v'}{aFu} + \frac{h'}{a\phi^2 Dw} = \frac{2\phi'w'}{aDh'} + O\left(\frac{fa^3}{Dr^6}\right).$$

Summing (2.15), (2.16) and (2.17), comparing with (2.14), we have

$$(2.18) \quad \begin{aligned} & H_0 - H \\ &= \left( \frac{4\phi'w'}{aDh'} + \frac{2\phi''w}{aDh'} - \frac{2\phi'wh''}{aDh'^2} \right) - 4\phi^{-3}n(\phi) + O\left(\frac{f^2a^2w}{D^3r^5}\right) + O\left(\frac{fa^3}{Dr^6}\right). \end{aligned}$$

Note that by Lemma 2.2,

$$H_0 - H = O\left(\frac{fa}{Dr^3}\right) + O\left(\frac{f^2a^2w}{D^3r^5}\right).$$

We claim that

$$(2.19) \quad \lim_{a \rightarrow \infty} \int_{S_a} (H_0 - H) d\sigma = \lim_{a \rightarrow \infty} \int_{S_a} (H_0 - H) d\sigma_0.$$

Noting that  $d\sigma - d\sigma_0 = O(r^{-1})d\sigma_0$ , it suffices to show that

$$(2.20) \quad \lim_{a \rightarrow \infty} \int_{S_a} O\left(\frac{fa}{Dr^3}\right) O(r^{-1}) d\sigma_0 = O(a^{-1})$$

and

$$(2.21) \quad \int_{S_a} O\left(\frac{f^2a^2w}{D^3r^5}\right) O(r^{-1}) d\sigma_0 = O(a^{-1}),$$

which in turn is implied by the stronger result

$$(2.22) \quad \int_{S_a} O\left(\frac{f^2a^2w}{D^3r^5}\right) d\sigma_0 = O(a^{-1}).$$



To prove (2.20). Since  $d\sigma_0 = a^2 Dw d\varphi d\theta$ , let  $f^2 = 1 + \alpha^2$ , consider

$$\begin{aligned}
0 &\leq \int_{S_a} \frac{fa}{Dr^4} d\sigma_0 = \frac{2\pi}{a} \int_0^l \frac{fw}{(w^2 + f^2 z^2)^2} d\varphi \\
&\leq \frac{2\pi}{a} \int_0^l \frac{(1+\alpha)w}{(w^2 + z^2 + \alpha^2 z^2)^2} d\varphi \\
&\leq \frac{2\pi}{a} \left( l + \int_0^l \frac{\alpha w}{(C_1^2 + \alpha^2 z^2)^2} d\varphi \right) \\
&= \frac{2\pi}{a} \left( l + \frac{1}{C_1^4} \int_{s(0)}^{s(l)} \frac{\alpha w/z'}{\left(1 + \frac{\alpha^2 z^2}{C_1^2}\right)^2} ds \right) \\
&\leq \frac{2\pi}{a} \left( l + \frac{C}{C_1^4} \int_{s(0)}^{s(l)} \frac{\alpha}{\left(1 + \frac{\alpha^2 z^2}{C_1^2}\right)^2} ds \right) \\
&\leq \frac{2\pi}{a} \left( l + \frac{C}{C_1^3} \int_{y_1}^{y_2} \frac{1}{(1+y^2)^2} dy \right) \\
&\leq \frac{2\pi}{a} \left( l + \frac{C\pi}{C_1^3} \right)
\end{aligned}$$

where we have used the fact that  $\frac{w}{z'}$  can be extended to a continuous function on  $[0, l]$  which is bounded by  $C$ . For (2.22), using similar computations, we have

$$\begin{aligned}
0 &\leq \int_{S_a} \frac{f^2 a^2 w}{D^3 r^5} d\sigma_0 \leq \frac{C_3}{a} + \frac{C_3}{a} \int_0^l \frac{\alpha^2 w^2}{(1 + \alpha^2 z'^2)(1 + \frac{\alpha^2 z^2}{C_1^2})^{5/2}} d\varphi \\
&\leq \frac{C_3}{a} + \frac{C_3}{a} \int_0^l \frac{\alpha^2 w^2}{(2\alpha z')(1 + \frac{\alpha^2 z^2}{C_1^2})^{5/2}} d\varphi \\
&\leq \frac{C_3}{a} + \frac{C_3 C}{2a} \int_0^l \frac{\alpha w}{(1 + \frac{\alpha^2 z^2}{C_1^2})^{5/2}} d\varphi \\
&\leq \frac{C_4}{a}.
\end{aligned}$$

Hence (2.19) is true:

$$\lim_{a \rightarrow \infty} \int_{S_a} (H_0 - H) d\sigma = \lim_{a \rightarrow \infty} \int_{S_a} (H_0 - H) d\sigma_0.$$

Next, by (2.18)

$$\begin{aligned} H_0 - H &= \left( \frac{4\phi'w'}{aDh'} + \frac{2\phi''w}{aDh'} - \frac{2\phi'wh''}{aDh'^2} \right) - 4\phi^{-3}n(\phi) + O\left(\frac{f^2a^2w}{D^3r^5}\right) + O\left(\frac{fa^3}{Dr^6}\right). \end{aligned}$$

Consider

$$\begin{aligned} &\int_{S_a} \left( \frac{4\phi'w'}{aDh'} + \frac{2\phi''w}{aDh'} - \frac{2\phi'wh''}{aDh'^2} \right) d\sigma_0 \\ (2.23) \quad &= 2\pi a \int_0^\pi \left( \frac{4\phi'ww'}{h'} + \frac{2\phi''w^2}{h'} - \frac{2\phi'w^2h''}{h'^2} \right) d\varphi \\ &= 2\pi a \int_0^\pi \frac{d}{d\varphi} \left( \frac{2\phi'w^2}{h'} \right) d\varphi \\ &= 0. \end{aligned}$$

So by (2.20) and (2.21), we have

$$\begin{aligned} (2.24) \quad \frac{1}{8\pi} \int_{S_a} (H_0 - H) d\sigma_0 &= -\frac{1}{2\pi} \int_{S_a} \phi^{-3}n(\phi) d\sigma_0 + O(a^{-1}) \\ &= \frac{1}{4\pi} \int_{S_a} n(\phi^{-2}) d\sigma_0 + O(a^{-1}). \end{aligned}$$

For each  $a$ , choose  $\partial B_a$  which is a Euclidean coordinate sphere enclosing  $S_a$  and let  $\Omega_a$  be the region between  $\partial B_a$  and  $S_a$ . The ADM mass of  $N$  is defined as

$$m_{ADM} = \lim_{a \rightarrow \infty} \frac{1}{16\pi} \int_{\partial B_a} (g_{ij,i} - g_{ii,j}) n^j d\sigma_0 = - \lim_{a \rightarrow \infty} \frac{1}{2\pi} \int_{\partial B_a} \phi^3 n(\phi) d\sigma_0$$

where  $n$  is the unit outward normal of  $\partial B_a$  with respect to  $\delta$ . As  $n(\phi) = O(r^{-2})$ ,

$$\phi^3 n(\phi) = n(\phi) + O(r^{-3}).$$

Clearly

$$\lim_{a \rightarrow \infty} \int_{\partial B_a} O(r^{-3}) d\sigma_0 = 0,$$

so

$$(2.25) \quad m_{ADM} = - \lim_{a \rightarrow \infty} \frac{1}{2\pi} \int_{\partial B_a} n(\phi) d\sigma_0.$$

By divergence theorem and the fact that  $\Delta\phi = 0$ ,

$$\begin{aligned}
\frac{1}{4\pi} \int_{S_a} n(\phi^{-2}) d\sigma_0 &= \frac{1}{4\pi} \int_{\partial B_a} n(\phi^{-2}) d\sigma_0 - \frac{1}{4\pi} \int_{\Omega_a} \Delta(\phi^{-2}) dV_0 \\
&= \frac{1}{4\pi} \int_{\partial B_a} n(\phi^{-2}) d\sigma_0 - \frac{1}{4\pi} \int_{\Omega_a} 6|\nabla\phi|^2 \\
&= \frac{1}{4\pi} \int_{\partial B_a} n(\phi^{-2}) d\sigma_0 + \int_{\Omega_a} O(r^{-4}) dV_0 \\
&= -\frac{1}{2\pi} \int_{\partial B_a} n(\phi) d\sigma_0 + \int_{\partial B_a} O(r^{-3}) d\sigma_0 + O(a^{-1}) \\
&= -\frac{1}{2\pi} \int_{\partial B_a} n(\phi) d\sigma_0 + O(a^{-1}).
\end{aligned}$$

So by (2.24) and (2.25), we have

$$\lim_{a \rightarrow \infty} \frac{1}{8\pi} \int_{S_a} (H_0 - H) d\sigma_0 = m_{ADM}.$$

Since

$$\lim_{a \rightarrow \infty} \frac{1}{8\pi} \int_{S_a} (H_0 - H) d\sigma = \lim_{a \rightarrow \infty} \frac{1}{8\pi} \int_{S_a} (H_0 - H) d\sigma_0,$$

we can conclude that

$$\lim_{a \rightarrow \infty} m_{BY}(S_a) = m_{ADM}(N, g).$$

This completes the proof of our result.  $\square$

### 3. PROOF OF COROLLARY 1.1

First of all, we have the following:

**Lemma 3.1.** *The Gaussian curvature  $K(\delta)$  of  $S_a$  with metric induced by  $\delta$  is positive.*

*Proof.* Let  $d\bar{s}^2$  be the metric on  $S_a$  induced by  $\delta$ . The Gaussian curvature of the revolution surface given by (1.5) is  $K_0 = -\frac{w''}{w} > 0$ . On the other hand,

$$(3.1) \quad d\bar{s}^2 = a^2((w'^2 + f^2 z'^2) d\varphi^2 + w^2 d\theta^2) = \bar{E} d\varphi^2 + \bar{G} d\theta^2.$$

The Gaussian curvature of  $d\bar{s}^2$  is then given by

(3.2)

$$\begin{aligned}
K(\delta) &= -\frac{1}{2\sqrt{\overline{EG}}} \left( \left( \frac{\overline{E}_\theta}{\sqrt{\overline{EG}}} \right)_\theta + \left( \frac{\overline{G}_\varphi}{\sqrt{\overline{EG}}} \right)_\varphi \right) \\
&= -\frac{1}{a^2 w \sqrt{w'^2 + h'^2}} \left( \frac{w'}{\sqrt{w'^2 + f^2 z'^2}} \right)' \\
&= -\frac{1}{a^2 w \sqrt{w'^2 + h'^2}} \left( \frac{w''}{\sqrt{w'^2 + f^2 z'^2}} - \frac{w'(w'w'' + f^2 z'z'')}{(w'^2 + f^2 z'^2)^{\frac{3}{2}}} \right) \\
&= -\frac{1}{a^2 w \sqrt{w'^2 + h'^2}} \left( \frac{w''}{\sqrt{w'^2 + f^2 z'^2}} - \frac{w'(w'w'' - f^2 w'w'')}{(w'^2 + f^2 z'^2)^{\frac{3}{2}}} \right) \\
&= \frac{-w''f^2}{a^2 w (w'^2 + f^2 z'^2)^2} \\
&= \frac{K_0 f^2}{a^2 D^4} > 0,
\end{aligned}$$

where we have used  $w'w'' + z'z'' = 0$ . □

*Proof of Corollary 1.1.* By Theorem 1.1, we just need to show that  $g|_{S_a}$  has positive Gaussian curvature as  $a \gg 1$ . By abuse of notations, we denote  $\delta|_{S_a}$  simply by  $\delta$  and  $g|_{S_a}$  by  $g$ . Noting that  $g = \phi^4 \delta$ . Similar to (2.4) in [16] or (2.14) in [10], we have

$$K(g) = \phi^{-4}(K(\delta) - 2\Delta_S(\log \phi))$$

where  $\Delta_S$  is the Laplacian on  $(S_a, \delta|_{S_a})$ . We have the following formula:

$$\Delta_S \psi = \Delta_{\mathbb{R}^3} \psi - \nabla_{\mathbb{R}^3}^2 \psi(n, n) - \overline{H}n(\psi)$$

where  $\overline{H}$  and  $n$  are the mean curvature and the unit outward normal vector of  $S_a$  with respect to  $\delta$  respectively. Letting  $\psi = \log \phi$ , we have

$$K(g) = \phi^{-4}(K(\delta) - 2\Delta_{\mathbb{R}^3} \psi + 2\nabla_{\mathbb{R}^3}^2 \psi(n, n) + 2\overline{H}n(\psi)).$$

Since  $\Delta_{\mathbb{R}^3} \phi = 0$ , we have

$$\Delta_{\mathbb{R}^3} \psi = \frac{\Delta_{\mathbb{R}^3} \phi}{\phi} - \frac{|\nabla \phi|^2}{\phi^2} = -\frac{|\nabla \phi|^2}{\phi^2} \leq 0.$$

By direct calculations, in Euclidean coordinates, for  $\psi = \psi(r)$ , we have

$$(\nabla_{\mathbb{R}^3}^2 \psi)_{ij} = \psi'' \frac{x^i x^j}{r^2} + \psi' \frac{\delta_{ij}}{r} - \psi' \frac{x^i x^j}{r^3}.$$

Therefore

$$\nabla_{\mathbb{R}^3}^3 \psi(n, n) = \frac{\psi''}{r^2} \langle X, n \rangle^2 + \frac{\psi'}{r} - \frac{\psi'}{r^3} \langle X, n \rangle^2$$

where  $X$  is the position vector. For  $\psi = \log \phi = \log(1 + \frac{m}{2r})$ , it is easy to see that the negative part of  $\nabla_{\mathbb{R}^3}^2 \psi(n, n)$  is of the order  $O(\frac{1}{r^3})$ :

$$(\nabla_{\mathbb{R}^3}^2 \psi(n, n))_- = O\left(\frac{1}{r^3}\right).$$

By Lemma 2.3,  $\overline{H} = O\left(\frac{f}{aD}\right)$ , thus

$$\overline{H}n(\psi) = O\left(\frac{f}{ar^2D}\right).$$

By (1.6), we have

$$\frac{1}{r^3} \leq \frac{f}{C_1 ar^2 D}.$$

We conclude that the negative part of  $-2\Delta_S(\log \phi)$  is of order  $O(\frac{f}{ar^2D})$ . Lemma 3.1 shows that  $K(\delta)$  is of order  $O(\frac{f^2}{a^2D^4})$ . As  $f \leq D$  and  $a \leq r$ , it is easy to see that  $K(\delta)$  dominates the negative part of  $-2\Delta_S(\log \phi)$  if  $\frac{f^2}{a} = o(1)$ . Hence the Gaussian curvature of  $S_a$  is positive as  $a$  large enough.  $\square$

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